

ADHM is Tachyon Condensation

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ABSTRACT: We completely realize the ADHM construction of instantons in D-brane language of tachyon condensations. Every step of the construction is given a physical interpretation in string theory, in a boundary state formalism valid all order in α' . Accordingly, equivalence between Yang-Mills configurations on D4-branes and D0-branes inside the D4-branes is proven, which shows that small instanton configurations of the Yang-Mills fields are protected against stringy α' corrections. We provide also D-brane realizations of the inverse ADHM construction, the completeness, and the noncommutative ADHM construction.

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1. Introduction: Beyond α'

Instantons are one of the most important nonperturbative effects in field theories, and the tremendous success of string theory in reproducing/predicting physical quantities in ordinary field theories partially owes to the fact that these nonperturbative effects have counterpart in string theory, as D-branes within (or intersecting with) D-branes. The most well-known example is the instantons in 4 dimensional Yang-Mills theory, which are considered to be equivalent to a bound state of D0-branes and D4-branes [1, 2]. But how rigorously this equivalence can be proven?

The equivalence, so far, has been supported by various consistency checks such as supersymmetries preserved, charges, masses and so on.* Precisely speaking, the instanton (or self-dual) gauge fields solve the equations of motion of Yang-Mills theory which is the D4-brane world volume theory at low energy. Thus the instanton

*For a review of the D0-D4 system and the relevance to the instantons, see [3]. Instanton configurations are obtained from the D0-D4 system in [4], but it is different from our standpoint: in [4] D0-branes were considered as a source for the Yang-Mills fields.

gauge configuration is trustable only for large size of the instantons with which the α' corrections are negligible. On the other hand, the D0-branes are point-like, and the description of them by the low energy effective action should be valid in a very different parameter region where the size of the instanton is sub-stringy.[†] Thus it is fair to say that we do not know what is the configuration in string theory corresponding to the low energy gauge theory instanton. It is possible that a configuration in a low energy effective theory would be extended (UV-completed) to a string theory configuration in many different ways. This problem is not only for the instanton \leftrightarrow D-brane equivalence but is rather generic in arguments based on BPS properties (although they are useful in various situations), and is very difficult to solve practically. Therefore it is quite interesting to connect these two descriptions, the instantons and the D0-branes, explicitly. In this paper, we are going to show the equivalence of these two pictures, beyond the α' corrections.

A clue is hidden in the famous ADHM (Atiyah, Drinfeld, Hitchin and Manin) construction of instantons [5], in which with ADHM data solving the ADHM equations an explicit self-dual gauge field can be constructed. Since the ADHM data have been identified with string excitations connecting the D0-branes and the D4-branes [2], and the ADHM equations can be seen as BPS equations of the low energy effective field theory on the D0-branes, the ADHM construction explicitly relates these two pictures.[‡] However this provides a further question, because the valid regions of the descriptions are very different.

In this paper, we will “derive” the ADHM construction from D-branes, and realize all the procedures of the ADHM construction in a D-brane setup rigorously, and thus provide physical meaning for each procedure. This implementation in terms of D-branes will be given in a boundary state formalism [6] (and a boundary string field theory (BSFT) [7, 8]), instead of the low energy effective actions. Thus the “derived” ADHM procedures are valid beyond stringy α' corrections, which resolves the question above. In other words, on D-branes the ADHM construction works regardless of the parameter regions in concern. And, this shows the equivalence of the two descriptions at all order in α' .

Let us explain briefly how we derive the ADHM construction in string theory which is valid at all order in α' . In string theory, different dimensional D-branes can be related via a K-theoretic argument [9] in which any kind of D-branes can be obtained by a single kind of D-branes by *tachyon condensation*. This is the D-

[†]The moduli space of a single instanton in SU(2) Yang-Mills theory consists of a real half-line specifying the instanton size ρ (and \mathbf{R}^4 giving the location of the instanton). The hypermultiplet appearing in the D0-brane effective action is related to the size as $S = \rho/\alpha'$, which has mass dimension one. Thus the effective theory is valid for $\alpha' S^2 \ll 1$, meaning $\rho \ll \sqrt{\alpha'}$.

[‡]Note that the instanton equation and the ADHM equation are scale-invariant (in the commutative space-time). Thus the moduli spaces near $\rho \ll \sqrt{\alpha'}$ and near $\rho \gg \sqrt{\alpha'}$ are of the same form, though these two regions are separated far way from each other.

brane descent [10] / ascent [11] relations. For example, in two pairs of a D4-brane and an anti-D4-brane, condensation of the tachyon whose profile is linear in the worldvolume coordinate leads to a single D0-brane [12, 13] (Atiyah-Bott-Shapiro construction [9, 8]). In this way, the D0-branes can be viewed as D4-branes, and along the way to include precisely the instantons into this scheme of the tachyon condensation, surprisingly we find that the ADHM construction naturally emerges. Therefore, *ADHM construction is nothing but a tachyon condensation*.

There have been attempts to realize the ADHM construction in D-branes [14, 15, 16], but our rigorous equivalence provides not only the direct relationship but also the following byproducts. The inverse ADHM construction, with which for a given instanton configuration the ADHM data is reproduced, can be derived in a similar manner as an ascent relation of the tachyon condensation. Furthermore, in the (inverse) ADHM construction, the completeness and the uniqueness of the ADHM construction have been shown [17]. This completeness can be lifted to the D-brane language, which even provides a simple proof of the completeness. We can “deconstruct” any D-brane system by infinitely many lowest dimensional D-branes and anti-D-branes.[§] Thus there is a unified picture for any D-brane system. This underlies the realization of the completeness and the Nahm construction of monopoles [20] for which we gave a stringy realization in our previous paper [21].

The organization of this paper is as follows. In Sec. 2, after a review of the ADHM construction and the D0-D4 system, we explain our idea of realizing the ADHM construction as a tachyon condensation. Then the detailed proof is provided in Sec. 3, with a stringy derivation of the noncommutative ADHM construction [22]. Sec. 4 is for the derivation of the inverse ADHM construction and the completeness. In Sec. 5, according to the realization of the ADHM construction provided in this paper, we give a conjecture stating that *the self-dual Yang-Mills configuration with arbitrary size solves non-Abelian Born-Infeld equations of motion obtained in string theory to all order in α' including derivative corrections*.[¶] Seiberg and Witten [23] argued this from the viewpoint of worldsheet supersymmetries. Discussions in Sec. 5 are on

[§]Instead of this, we can use a higher dimensional one, but the lowest dimensional D-branes may be the simplest to study [18, 19].

[¶]We show in this paper that the D0-D4 system possessing the ADHM data without the constraint (the ADHM equation) provides the gauge fields on the D4-brane where the gauge field configurations are computed by the ADHM construction. Therefore the equivalence is shown at off-shell. (Precisely speaking, we need to require a weak condition on the asymptotic behavior of the Dirac operators, but don’t need the ADHM equation itself in showing the equivalence.) In this sense the off-shell ADHM construction works in string theory. Our result for on-shell configurations is at small instanton singularity, which strongly supports this conjecture concerning arbitrary points in the instanton moduli space. We note that similar statements have been put within the context of “Born-Infeld” corrections [24], but at the best of our knowledge no statement including all the derivative corrections has been made. (This might be subtle, in the sense that in the non-Abelian case the “Born-Infeld corrections” may not make sense because they can be traded with commutators of covariant derivatives and the notion of “constant” field strength is not well-defined.)

various low-energy limits, the Atiyah-Singer index theorem [25], and generalizations of the ADHM construction.

2. Tachyon Condensation and ADHM

2.1 Review: ADHM construction of instantons and D0-D4 system

Before explaining our strategy to derive the ADHM (and the inverse ADHM) construction of instantons from the tachyon condensation of unstable D-branes, we briefly summarize the ADHM construction itself and corresponding D-brane configurations in superstring theory.

The ADHM construction is a powerful tool for constructing gauge configurations of instantons explicitly. For the construction of k instanton configurations in $SU(N)$ Yang-Mills theory, we need the following ADHM data: S which is an $N \times 2k$ complex constant matrix and $X_\mu (\mu = 1, 2, 3, 4)$ which are hermitian $k \times k$ matrices. Then the procedures of the ADHM construction starts with finding N zeromodes of a zero dimensional “Dirac operator” ∇^\dagger ,

$$\nabla^\dagger V = 0, \quad \nabla^\dagger \equiv \left(\overbrace{S^\dagger}^N \quad \overbrace{e_\mu^\dagger \otimes (x^\mu \mathbb{I}_k - X^\mu)}^{2k} \right) \Big\}_{2k}. \quad (2.1)$$

Here e_μ ($\mu = 1, 2, 3, 4$) are a representation of quaternion, $e_\mu \equiv (i\sigma_i, \mathbb{I}_2)$, where σ_i ($i = 1, 2, 3$) are Pauli matrices. Arraying the N independent zeromodes constitutes V which is a $(N+2k) \times N$ matrix normalized as $V^\dagger V = \mathbb{I}_N$. This V is a function of x^μ through the Dirac operator ∇^\dagger , and the desired instanton gauge field configuration is given by the formula

$$A_\mu = V^\dagger \partial_\mu V. \quad (2.2)$$

For this gauge field to be self-dual, the ADHM data should satisfy the ADHM equations

$$\text{Tr} [\sigma_i (S^\dagger S + ((e^\dagger)^\mu e^\nu X^\mu X^\nu))] = 0 \quad (i = 1, 2, 3) \quad (2.3)$$

In brane language, this system of instantons in Euclidean 4 dimensional Yang-Mills theory has been known to be described by a combined brane configuration of k D0-branes and N D4-branes in type IIA superstring theory [2]. The low energy effective field theory on the N D4-branes is the $1 + 4$ dimensional $U(N)$ Yang-Mills theory with maximal supersymmetries. If we restrict our attention to the gauge fields with spatial indices A_μ ($\mu = 1, 2, 3, 4$), then the instanton configurations equivalent to self-dual configurations of the gauge fields are compatible with the BPS condition of preserving half of the supersymmetries on the worldvolume. The instanton charge is shown to be equal to the total D0-brane charge bound on the D4-brane, through the

Ramond-Ramond coupling in the D4-brane action. Therefore, Yang-Mills instantons have the same amount of charges, masses and supersymmetries, as those of the D0-branes on the D4-branes.

On the other hand, one can look at this brane system from the viewpoint of the worldvolume effective field theory on the k D0-branes. The low energy matter content includes scalar field excitations X^μ from strings connecting the D0-branes, and another scalar field S connecting the D0-branes and the D4-branes. The Chan-Paton factor suggests that X^μ are $N \times N$ hermitian matrices, and S is a complex $N \times k$ matrix tensored with an $SU(2)$ vector index (this $SU(2)$ is one of $SU(2) \times SU(2) \sim SO(4)$ which is the worldvolume Lorentz symmetry of the D4-branes and should be seen by the D0-branes as a global symmetry, *i.e.* the R-symmetry). This set (S, X^μ) should be identified with the ADHM data [2], and in fact, the BPS condition for (S, X^μ) is equivalent to the ADHM equation (2.3).

These facts show that D-brane techniques are quite powerful in that a part of the ingredients of the ADHM construction already appear as matter contents and supersymmetry conditions on the worldvolumes. Furthermore, introduction of small D-brane probes [14] enables one to actually realize the ADHM construction explained above. In a D5-D9 system which is T-duality equivalent to the above D0-D4 system, one introduces a probe D1-brane whose effective worldvolume sigma model realizes the ADHM formula (2.2) for the background gauge fields A_μ on the D9-branes. This probe analysis was generalized to the Nahm construction of monopole [15] and the Nahm transformation for the gauge fields on T^4 by using the T-duality [16].

However, these interesting connections to the D-branes introduce probes, which means that one can get only the information seen by the probes. Furthermore, the descriptions use effective actions on the probe and so valid only in the low energy limits. Therefore, the probe method is not enough to show that in fact the two descriptions, one by the D4-branes (self-dual equations) and one by the D0-branes (ADHM equations) are completely equivalent beyond the stringy corrections. Another important point missing in the probe method is the inverse ADHM construction and the completeness [17]. The ADHM construction gives all the instanton configurations up to gauge transformations, that is the completeness and the uniqueness of the construction. This was shown explicitly [17] by applying the inverse ADHM construction to the gauge fields constructed by the ADHM construction.^{||}

In the following, we present a complete derivation of the ADHM and the inverse ADHM constructions, without using any probes, and in all order in stringy corrections. This is possible owing to an exact treatment of the tachyon condensation in the BSFT and the boundary state formalism.

^{||}The realization of the Nahm transform [16] is showing the completeness, but there the world-volume is compactified and resultantly the information on the S field is unclear.

2.2 Our idea: brane configurations and tachyon condensation

The powerfulness of the ADHM construction is due to the difference in dimensions to solve. The ADHM equation is a purely algebraic equation while the instanton equation is a partial differential equation which is highly nontrivial. It is miraculous that those two are equivalent. However, this miracle is shared by generic D-brane physics — there is a notion called “brane democracy” first mentioned by Townsend [26] which is generalized to mean that through various dualities any dimensional branes may play central role in constructing the full string/M theory and in revealing dynamics of any other dimensional branes. One noble example is Matrix theory [27] in which lowest dimensional D-branes are constituents to build higher dimensional M-theory physics. A shortcoming of the Matrix theory is that charges of the constituents remains in any setup made out of them, but it has been overcome by K-matrix theory [18, 19] in which the constituents are unstable D-branes and without the restriction of the charges one can truly construct any brane configurations out of them through tachyon condensation [10], the annihilation of unstable D-branes, developed by Sen.

The brane configuration of our concern consists of two different kinds of D-branes, the k D0-branes and the N D4-branes. In the sense described above, it is natural to consider a treatment of this system in terms of a single kind of D-branes. There are two ways to realize this:

- (a) By D4-branes solely. One can represent the k D0-branes by a tachyon condensation of $2k$ pairs of D4-branes and anti-D4-branes. This is a D-brane descent relation. In total, one has $N + 2k$ D4-branes and $2k$ anti-D4-branes.
- (b) By D0-branes only. The N D4-branes can be constructed by a tachyon condensation of infinite number of pairs of D0-branes and anti-D0-branes. This is called a D-brane ascent relation, found in [11] and developed in [18, 19].

It turns out that all of the ADHM construction and the inverse ADHM construction are realized in these two ways of understanding of the brane configurations. In fact, *the representation (a) realizes the ADHM construction, while the representation (b) is nothing but the inverse ADHM construction.* A schematic picture for the ADHM construction (a) is shown in Fig. 1, and for the inverse ADHM construction (b) in Fig. 2.

Let us look at the equality (a) more closely. As we shall see in the next section, the tachyon field, arising from the string connecting the D4-branes and the anti-D4-branes, has a peculiar form to incorporate the D0-D0 string excitations (X^μ) and the D0-D4 string excitations (S) after the tachyon condensation. Interestingly, an exact treatment of this leads to the form

$$T = \lim_{u \rightarrow \infty} u \nabla^\dagger . \quad (2.4)$$

In other words, *the Dirac operator is the tachyon*.^{*} The physical essence of the tachyon condensation is that, once the tachyon expectation value becomes infinite, the corresponding pair of the D4 and anti-D4-branes disappear. Therefore, from the relation (2.4), the D4-branes surviving after the tachyon condensation is identified with the zeromodes of the Dirac operator [29].[†] V in (2.1) is interpreted as a “wave function” of the remaining D-branes. One can view this procedure just as a change of basis of the Chan-Paton factor, and because the basis now depends on x , there appears a nontrivial connection on the remaining D4-branes, which is the gauge field A_μ given by the seemingly-unitary transformation of a trivial connection, (2.2). In the next section we make this statement more precise and explain its relation to a Berry’s phase on the worldsheet description of strings in target space background fields. Fig. 1 shows these processes schematically.

The representation (b) gives in a similar manner the inverse ADHM construction. Here again, the tachyon profile coming from the strings connecting the infinite number of pairs of the D0-branes and the anti-D0-branes is found to be identical to the Dirac operator in Euclidean 4 dimensions which is a necessary ingredient of the inverse ADHM construction. See Fig. 2. Because we need infinite number of pairs of D0-branes and anti-D0-branes, the tachyon is an infinite dimensional matrix, which turns out to be a matrix-representation of the Dirac operator $e_\mu^\dagger(\partial_\mu + A_\mu(x))$.

A new outcome of our method using D-branes is concerning the completeness. Corrigan and Goddard showed explicitly [17] that performing the ADHM and the inverse ADHM constructions succeeding ends up with going back to the original configuration, which shows the completeness and the uniqueness. We find a more direct way of checking the completeness, without using explicitly the relations (a) and (b): in Sec. 4.2 we show that there is a direct relation between the D4-brane descriptions and the D0-brane description, which is horizontal arrow in Fig. 3.

A surprise is that our “derivation” of the ADHM construction using the tachyon condensation on the D4-anti-D4 system turns out to be a realization of the original derivation of the ADHM construction [5]. As well-phrased in Atiyah’s lecture note [31], the instanton gauge field in the ADHM construction is given as an induced connection on a subspace of a trivial vector bundle over $S^4 = P_1(\mathbf{H})$. This $P_1(\mathbf{H})$ is a quaternion projective line defined by homogeneous coordinates (x, y) with $x, y \in \mathbf{H}$ and identified as $(x, y) \sim (xq, yq)$. The projection onto the sub-bundle is given by a map $v(x, y) = Cx + Dy$ which is a $(k + N) \times N$ matrix of quaternions with constant matrices C and D . More precisely, the operator $\mathbb{I}_{k+N} - vv^*$ is the projection onto the sub-bundle of our concern. A certain constraints on C and D ends up with (anti-)self-dual connections on the sub-bundle, which is the essence of the ADHM construction. The parameters C and D become the ADHM data S and X^μ after redundant degrees of freedom are gauged away. In addition, we can choose a gauge $y = 1$ in the

^{*}Relations between tachyons and Dirac operators have been discussed in [16, 28].

[†]See also [30].

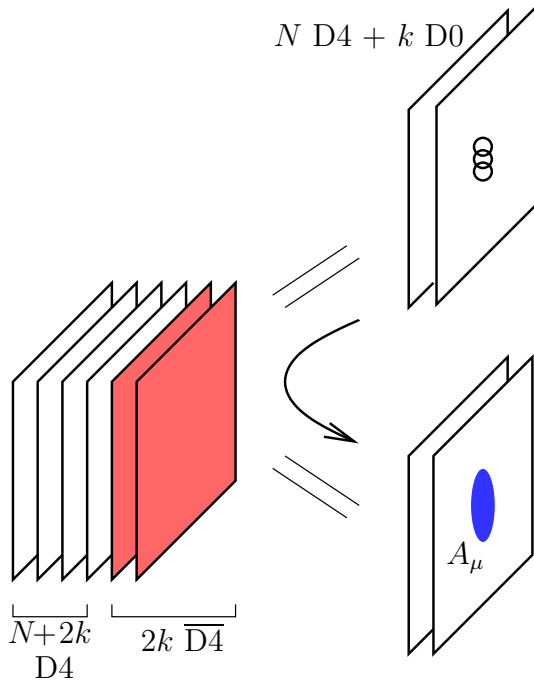


Figure 1: D-brane realization of the ADHM construction via a tachyon condensation. The ADHM data on the D4-D0 system (upper-right) is directly translated to the tachyon profile on the D4-anti-D4 system (left). After the tachyon condensation, it is shown to be equivalent to just D4-branes with nontrivial gauge field (lower-right).

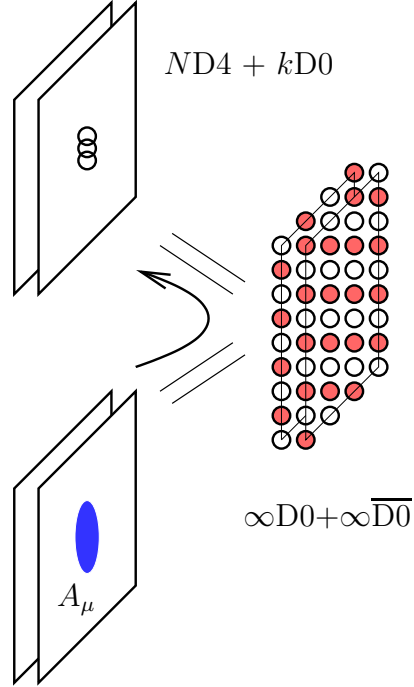


Figure 2: The idea of the inverse ADHM construction. The instanton configuration of the gauge field A_μ (lower-left) is directly encoded in the tachyon profile of the ∞ number of the D0-anti-D0-branes (right). In a basis proper for the D0-branes, we can read the information of the D0-branes and D0-D4 strings which are the ADHM data (upper-left).

representation of $P_1(\mathbf{H})$. Then, we find that the linear matrix function v is realized by our tachyon configuration, and the projection is given a physical interpretation that zeros of the tachyon correspond to “wave functions” (Chan-Paton factors) of the surviving D-branes. The extended space \mathbf{H}^{k+N} with the trivial bundle is nothing but the vector space of the Chan-Paton factor of the brane-anti-brane system.[‡] The tachyon condensation singles out the sub-bundle with induced connections on it.

In our derivation, we haven’t referred to any on-shell condition of the fields appearing and thus to any self-dual equations (except a condition on the number of the Dirac zero-modes). In this sense our construction works even off-shell.[§] The correspondence between the ADHM data satisfying the ADHM equation (2.3) and

[‡]Atiyah’s case [31] treats \mathbf{H}^{N+k} resulting in a gauge group $\mathrm{Sp}(N)$ while in our case the vector space is \mathbf{C}^{N+2k} for the gauge group $\mathrm{SU}(N)$.

[§]At off-shell, often the boundary state might suffer from divergence and not well-defined, but our treatment can be justified by the BSFT.

the (anti-)self-dual configuration of A_μ appears once we impose the supersymmetry condition on both sides.

3. Derivation of ADHM by D-branes

3.1 Derivation

As briefly described in the previous section, we are interested in viewing the D0-D4 system solely by D4-branes, by replacing the D0-branes with pairs of D4-branes and anti-D4-branes accompanied by the tachyon condensation (the relation (a) in Sec. 2.2). Eventually this derives the ADHM construction of instantons, as we shall see. The way we look at the D0-D4 system helps to describe it rigorously in terms of a boundary state. When D-branes with different dimensionalities are present, there is a complication in writing a boundary states of that system because of possible twist operations (changing boundary conditions) on the boundaries of the string worldsheet. However if one lifts the D0-branes to the pairs of D4-branes and anti-D4-branes with the tachyon condensation, this complication disappears, which is another motivation for our description with the tachyons.

The charges of the k instantons in $SU(N)$ Yang-Mills theory is provided by k D0-branes residing on coincident N D4-branes. To be precise, this correspondence is valid when the gauge field configuration is at the small instanton singularity. Then the location of the D0-brane on the D4-brane worldvolume is identified with the point-like location of the instantons. To obtain instantons with finite size, one has to let the massless mode of the D0-D4 strings condensate, and roughly speaking, the expectation value of this massless field on the D0-branes is the size of the instanton.

Let us consider first the zero size instantons, equivalently neglecting the D0-D4 strings. A D0-brane can be described by the following tachyon condensation on two pairs of a parallel D4-brane and an anti-D4-brane,

$$t = ux^\mu e_\mu^\dagger, \quad (3.1)$$

with $u \rightarrow \infty$ limit. This is called Atiyah-Bott-Shapiro construction [9, 8], and in the limit $u \rightarrow \infty$ this configuration becomes a solution of a boundary string field theory [12, 13], and thus is on-shell and a consistent background of string theory. To have k D0-branes, we prepare $2k$ pairs of D4-anti-D4-branes. The location of the D0-branes is encoded as zeros of the tachyon profile, so to introduce generic location of the k D0-branes, we generalize (3.1) to

$$t = u(x^\mu \mathbb{I}_k - X^\mu) \otimes e_\mu^\dagger \quad (3.2)$$

where X^μ are $k \times k$ constant hermitian matrices. When X^μ are simultaneously diagonalizable, it is clear that this gives the location of the D0-branes after the tachyon condensation. Even when they are not, it has been shown that this incorporation

(3.2) of the X^μ matrices with the D4-anti-D4-brane boundary state results in a D0-brane boundary state with transverse scalar field profile X^μ [19], thus (3.2) is the correct profile including the massless excitation of the D0-D0 strings.

The total system of our concern consists of $N + 2k$ D4-branes and $2k$ anti D4-branes, thus the tachyon T in the system is a complex $2k \times (N + 2k)$ matrix. The system has the gauge invariance $U(N + 2k) \times U(2k)$, and the tachyon is in a bi-fundamental representation with respect to this gauge symmetry. The low-lying excitations of the strings also include the gauge fields on the D4-branes and the anti-D4-branes, $A_\mu^{\text{D4}}(x)$ and $A_\mu^{\text{antiD4}}(x)$. We put them vanishing, $A_\mu^{\text{D4}}(x) = A_\mu^{\text{antiD4}}(x) = 0$. These low-lying excitations of the D4-anti-D4-branes can be conveniently written as an $(N + 4k) \times (N + 4k)$ matrix,

$$M = \left(\begin{array}{c|c} \overbrace{A^{\text{D4}}}^{N+2k} & \overbrace{T^\dagger}^{2k} \\ \hline T & A^{\text{antiD4}} \end{array} \right) \left. \vphantom{\begin{array}{c|c} \overbrace{A^{\text{D4}}}^{N+2k} & \overbrace{T^\dagger}^{2k} \\ \hline T & A^{\text{antiD4}} \end{array}} \right\} \begin{array}{l} N+2k \\ 2k \end{array}, \quad (3.3)$$

which is known as a superconnection. Then the gauge symmetry $U(N + 2k) \times U(2k)$ acts as $M \rightarrow U^\dagger M U + U^\dagger dU$, $U = \text{diag}(U_1, U_2)$ where $U_1 \in U(N + 2k)$, $U_2 \in U(2k)$. For the present case, the previous tachyon t (3.2) is embedded in the tachyon T as

$$T = \left(\begin{array}{c|c} \overbrace{0}^N & \overbrace{t}^{2k} \\ \hline & \end{array} \right) \left. \vphantom{\begin{array}{c|c} \overbrace{0}^N & \overbrace{t}^{2k} \\ \hline & \end{array}} \right\} 2k. \quad (3.4)$$

where the entry “0” means a vanishing matrix of the size $2k \times N$.

In general, nothing prevents us from turning on this vanishing part of the tachyon matrix T . In fact, this part should correspond to the excitation of the string connecting the remaining N D4-branes and the created k D0-branes. This is obvious when we look at the matrix (3.3). The lower-right $4k \times 4k$ corner becomes the k D0-branes after the tachyon condensation $u \rightarrow \infty$, so the lower-left corner should represent the D0-D4 strings. Let us turn on generic value in the left half entries of the tachyon matrix T as

$$T = \lim_{u \rightarrow \infty} \left(\begin{array}{cc} uS^\dagger & t \end{array} \right). \quad (3.5)$$

where S is a constant complex $N \times 2k$ matrix.* The indices which S carries in fact coincide with that of the massless excitation of the strings connecting k D0-branes and N D4-branes: it is known that a field of fundamental representation in $U(k)$ appears from the D0-D4 strings, and it is charged under the “global” $U(N)$ as a fundamental and in the $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations of the “internal” rotation group $SO(4) \sim SU(2) \times SU(2)$ acting on the worldvolume of the D4-branes. At this stage, we established the equality of the upper-right and the left figures in Fig. 1.

*We expect X^μ dependent parts of S corresponds to the massive excitations.

Then with this nonzero S , which D-branes remain after the tachyon condensation? The answer to this question is another equality connecting the left and the lower-right figures in Fig. 1. In the BSFT, the pair of D4 and anti-D4-branes with infinite value of the tachyon vanishes, while if the tachyon remains zero those D-branes survive. Now the tachyon is a $2k \times (N + 2k)$ matrix, so one has to diagonalize the whole tachyon matrix by a gauge transformation of $U(N + 2k) \times U(2k)$,

$$T \rightarrow T' = U_2^\dagger T U_1 \quad (3.6)$$

where U_i is the gauge transformation associated with the gauge field $A^{(i)}$. We can use this gauge degrees of freedom to get the following canonical form of the tachyon,

$$T' = \lim_{u \rightarrow \infty} u \left(\begin{array}{c|c} \overbrace{\quad}^N & \overbrace{\quad}^{2k} \\ \hline 0 & \begin{matrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{matrix} \end{array} \right) \Bigg\}_{2k} \quad (3.7)$$

where the left half of the matrix is vanishing while the right half ($2k \times 2k$) is diagonal with nonzero entries.[†] Generically this form of the matrix is available.

In this rotated basis of the Chan-Paton factor, it is easy to figure out which brane is surviving in the $u \rightarrow \infty$ limit of the tachyon condensation. The D4-branes corresponding to the left half (column $1, \dots, N$) are surviving the annihilation process while the right half (column $N + 1, \dots, N + 2k$) will be pair-annihilated with the $2k$ anti-D4-branes.

Let us look at the properties of the remaining N D4-branes. Now according to the above gauge transformation U_1 and U_2 , we have actually a nonzero gauge field on the $(N + 2k)$ D4-branes,

$$A_\mu^{(1)} = U_1^\dagger \partial_\mu U_1 . \quad (3.8)$$

But what we need is only a part of this matrix, given by the ij components ($i, j = 1, \dots, N$), because other Chan-Paton indices are for disappearing D4-branes and unphysical. For the physical gauge field on the remaining N D4-branes, we need only a part of the information of the gauge rotation matrix U_1 . If we explicitly write the matrix U_1 as

$$U_1 = \left(\begin{array}{c|c} \overbrace{\quad}^N & \overbrace{\quad}^{2k} \\ \hline V & V' \end{array} \right) \Bigg\}_{N+2k} , \quad (3.9)$$

then our physical part ($N \times N$) of the gauge field is given by

$$[A_\mu^{(1)}]_{N \times N} = V^\dagger \partial_\mu V . \quad (3.10)$$

[†]This nondegeneracy condition is an assumption of the ADHM construction.

This (3.10) is nothing but the ADHM formula (2.2). We can conclude that we deduce the ADHM construction if this V is a collection of the normalized zeromodes of the Dirac operator (2.1). And this is in fact the case. First, the normalization condition required in the ADHM construction, $V^\dagger V = \mathbb{I}_N$, is just a part of the unitarity condition of U_1 . So this is satisfied. Second, the zeromode condition (2.1) is found to be just a part of the unitary rotation (3.6), because the Dirac operator is exactly the tachyon field T , and the rotated form of T' has zeros as in (3.7).

Here completes the derivation of the ADHM construction from a tachyon condensation in D-branes. We provide a rigorous proof in Sec. 3.3 by realizing the tachyon condensation procedure in the boundary state formalism. We will see that this relation between the data X^μ, S on the unstable D4-anti-D4-branes and the gauge fields A_μ after the tachyon condensation is exact, and thus those two descriptions are equivalent.

We used a unitary transformation for the rotation of the basis of the Chan-Paton factor, but this is equivalently described by just the notion of the zeromode eigen function of the tachyon matrix T , as originally described in [29]. In this terminology, the gauge field A_μ is provided as a Berry's phase as in our previous paper for the Nahm construction [21]: the zeromode eigen states ($V_i(x) = \langle x|V_i\rangle$, $i = 1, \dots, N$) are functions of x , and furthermore, x is a function of the worldsheet boundary time σ , therefore on the worldsheet action, the Berry's phase is induced,

$$\gamma_{ij} = \oint d\sigma \langle V_i | \partial_\sigma | V_j \rangle = \oint d\sigma \partial_\sigma X^\mu(\sigma) \langle V_i | \partial_\mu | V_j \rangle . \quad (3.11)$$

This is a worldsheet boundary coupling to a background gauge field given by the coefficients,

$$[A_\mu]_{ij} = \langle V_i | \partial_\mu | V_j \rangle , \quad (3.12)$$

which is the ADHM formula.

3.2 Noncommutative ADHM and identification of S

One of the recent interesting topics has been solitons on noncommutative spaces, which was initiated by Nekrasov and Schwarz [22] who related the resolution of the small instanton singularity in the ADHM moduli space with the noncommutativity. In [22], how the ADHM construction in the noncommutative space works was explained: the noncommutative ADHM construction is obtained simply by replacing all the procedures in the ADHM construction by their noncommutative generalization. The product is replaced with Moyal $*$ product, and the ADHM equation is modified to have a resolution of the singularity. In this subsection we derive the noncommutative ADHM construction, and explain why this works in this way, in terms of D-branes and the tachyon condensation.

As seen also in [22], the space noncommutativity is introduced as a background constant NS-NS B -field on the worldvolume of the D-branes [32, 23]. So let us think of putting all the brane setup in the background constant B -field. The background B -field effectively induces a constant field strength on the D-branes, $F_{\mu\nu} = B_{\mu\nu}/2\pi\alpha'$. In the language of the boundary state of the D-branes, this simply induces a term $\oint d\sigma F_{\mu\nu} x^\mu(\sigma) \dot{x}^\nu(\sigma)$ in the boundary action, and nothing more than that. We have to perform the Seiberg-Witten map [23] to obtain the description in terms of fields in the equivalent noncommutative space. This makes things complicated, and furthermore for finite α' there is no known explicit expression for the Seiberg-Witten map. That is to say, the elegant noncommutative ADHM construction, with just the noncommutative zero mode equations (2.1) and the noncommutative overlap (2.2), appear to be difficult to show up in this attempt.

Instead of this trivial trial, we take a different route to realize the noncommutativity, which turns out to lead us to the realization of the noncommutative ADHM construction. Consider a single D4-brane. Putting it in the constant B -field is equivalent on the worldvolume to regard the D4-brane as a bound state of infinitely many D0-branes [33, 34]. Note that this is possible without the $\alpha' \rightarrow 0$ limit [33]. This is a famous example of Matrix theory. The transverse scalars of the D0-branes are turned on as $\Phi_\mu = \hat{x}_\mu$ ($\mu = 1, 2, 3, 4$) where infinite dimensional matrices \hat{x}_μ satisfy the noncommutative algebra[‡]

$$[\hat{x}_1, \hat{x}_2] = i\theta_{12} = -i\alpha'/B_{12} \ , \quad [\hat{x}_3, \hat{x}_4] = i\theta_{34} = -i\alpha'/B_{34} \ . \quad (3.13)$$

This is the appearance of the noncommutativity. Here \hat{x}_μ are infinite dimensional matrices, and their explicit expression is given by

$$\begin{aligned} [\hat{x}_1 + i\hat{x}_2]_{(n_1, n_2), (m_1, m_2)} &= \sqrt{2\theta_{12}} \sqrt{n_1} \delta_{n_1, n_2-1} \delta_{m_1, m_2} \ , \\ [\hat{x}_3 + i\hat{x}_4]_{(n_1, n_2), (m_1, m_2)} &= \sqrt{2\theta_{34}} \sqrt{m_1} \delta_{n_1, n_2} \delta_{m_1, m_2-1} \ . \end{aligned} \quad (3.14)$$

Henceforth, we use this matrix representation for the $N + 2k$ D4-branes and $2k$ anti-D4-branes. We will find that this way of considering the noncommutativity leads to the noncommutative ADHM construction. Note that for the anti-D4-branes, we consider the same transverse scalar field configuration (3.13) of anti-D0-branes.

As a warm-up, we consider the example of a pair of 2 D4-branes and 2 anti-D4-branes. We know that the tachyon configuration on these pairs (3.1) produces a single D0-brane after the tachyon condensation $u \rightarrow \infty$. What about the case with the noncommutativities? The D4-branes consist of infinite number of D0-branes, while the anti-D4-branes are made of anti-D0-branes.[§] Because we have the D0-branes and

[‡]Note that this \hat{x} is different from that appearing in the usual quantum mechanics where the Heisenberg algebra $[\hat{x}_\mu, \hat{p}_\nu] = i\delta_{\mu, \nu}$ is satisfied. The latter will be used in Sec. 4.

[§]The total charge of the D0-branes is vanishing, as easily seen in the Ramond-Ramond coupling in the BSFT action of the brane-anti-brane [12, 13] which reads $\int C_0 \wedge (e^B - e^{\bar{B}}) = 0$.

the anti-D0-branes, there exists a complex tachyon field as an excitation of a string connecting those. It turns out that the tachyon profile (3.1) with replacement of x_μ with the matrix \hat{x}_μ ,

$$t = u(\hat{x}^\mu - X_\mu \mathbf{1}_\infty) e_\mu^\dagger \quad (3.15)$$

with the limit $u \rightarrow \infty$ is a solution of a BSFT. Here X_μ are constant parameters. To see this, we apply the idea of [29, 30] for the above tachyon profile (3.15). The zeromode of the above matrix specifies the remaining D0-brane. In fact, there exists a single zeromode given by

$$\begin{aligned} \Psi = & \exp \left[-\frac{X_1^2 + X_2^2}{4\theta_{12}} - \frac{X_3^2 + X_4^2}{4\theta_{34}} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & \otimes \left[\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{X_1 + iX_2}{\sqrt{2\theta_{12}}} \right)^n \right] \left[\sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \left(\frac{X_3 + iX_4}{\sqrt{2\theta_{34}}} \right)^m \right] |n, m\rangle \end{aligned}$$

where the first 2-vector is for the vector space on which e_μ^\dagger acts, and we have chosen the representation of the base vector space of the noncommutative operators as the standard one labelled by $|n, m\rangle$ ($n, m = 0, 1, 2, \dots$) (we need a tensor product of two Hilbert spaces since we are working in $4 = 2 + 2$ dimensions). We included the normalization factor already in Ψ .

The location of the surviving D0-brane can be found by the vacuum expectation value of \hat{x}_μ or in other words, the scalar field matrix element with the index given by the above Ψ , as

$$\Psi^\dagger \hat{x}_\mu \Psi = X_\mu, \quad (3.16)$$

as anticipated.

The generalization of the tachyon profile (3.1) to the noncommutative case is given by (3.15), therefore, the noncommutative generalization of the full tachyon operator (3.5) concerning the ADHM construction should be provided by replacing x_μ with by the infinite dimensional matrix \hat{x}_μ . The computation of finding zero-modes can be done in the infinite dimensional matrix multiplications, and this is nothing but working with Moyal \ast -product with usual x_μ . Thus we have derived the noncommutative ADHM construction.

In the previous subsection, we have identified a part of the tachyon matrix S as an excitation of the D0-D4 strings. There we presented an argument that this S carries a correct charge of the strings. Here we show that, for small fluctuation of S , this gives the mass spectrum identical with the fluctuation of the D0-D4 strings. An explicit instanton configuration in $U(2)$ noncommutative Yang-Mills theory was given in [35] via the noncommutative ADHM construction [22]. There explicit construction with the parameter S results in the following instanton configuration of the gauge

field (see Eqs. (5.10)–(5.12) of [36] where S is written as ρ):

$$\hat{x}_\mu - \theta_{\mu\nu} A_\nu = U_0^\dagger \hat{x}_\mu U_0 + \left[\hat{x}_\mu |0, 0\rangle \langle 0, 0| + |0, 0\rangle \langle 0, 0| \hat{x}_\mu \right] \otimes \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix} + \mathcal{O}(\rho^2) ,$$

where U_0 is a shift operator which shifts the Hilbert space index by one, $U_0|s\rangle = |s+1\rangle$ where $|s = (n+m)(n+m+1)/2 + m\rangle \equiv |n, m\rangle$. The first term in the right hand side of the above solution is the well-known noncommutative soliton generated by the shift operator [37]. The rest terms are a deviation from the shift-operator-generated noncommutative soliton, and they come in the first off-diagonal entries in this Hilbert space, as specified by $\hat{x}_\mu |0, 0\rangle \langle 0, 0|$ or $|0, 0\rangle \langle 0, 0| \hat{x}_\mu$. These entries are nothing but the ones giving the mass spectrum of the hypermultiplets coming from the D0-D4 strings, as shown in [38]. Therefore in this noncommutative example, through the ADHM construction, it is explicitly shown that the matrix S appearing in a part of the tachyon is in fact the D0-D4 string excitation. This also implies that the normalization of S in (3.5) is indeed correct.

3.3 Exactness shown in boundary state formalism

The derivation of the ADHM construction in terms of D-branes presented in Sec. 3.1 is just an analysis of the bases of the matrix-valued tachyon field. Nevertheless, the equivalence of the D0-D4 system and the D4-branes with instanton gauge fields can hold beyond the α' corrections, which we will show in this subsection. We show this by using a boundary state formalism intimately related to the BSFT. Boundary states are “states” in the closed string Hilbert space, specified by boundary conditions on the string worldsheet. The quantized world sheet scalar fields in the closed string picture, $\hat{X}^\mu(\sigma)$, act on them. The boundary state is one of the definitions of D-branes, thus once one can prove that two boundary states are equal, it immediately shows that those two D-branes are identical.

The action of a BSFT for brane-anti-branes was constructed in [12, 13] with worldsheet boundary interactions including tachyons. This is straightforwardly generalized to boundary states, whose useful expression can be found in K-matrix theory, [19].[¶] For simplicity, we ignore all the worldsheet fermions and ghosts which are not relevant for our purpose. Then the boundary state is given as

$$|B\rangle = \int [dx] e^{-S_b} |x\rangle . \quad (3.17)$$

The ket $|x\rangle$ is an eigenstate of the closed string worldsheet scalar coordinates \hat{X}^μ , $\hat{X}^\mu |x\rangle = x^\mu(\sigma) |x\rangle$, where σ parameterizes the boundary of the string worldsheet.

[¶]However, all of the situations considered in the literature (except [39]) have dealt with equal number of D-branes and anti-D-branes which makes it possible to trade the (gamma) matrices appearing in the boundary interaction for boundary fermions. But in our present case, since the number of D4-branes is different from that of the anti-D4-branes, we cannot use the fermion representation. Instead, we use the explicit matrix formula for the boundary interaction.

The boundary perturbation e^{-S_b} is represented as a partition function for a quantum mechanics with Hamiltonian $(M_0)^2$ acting on a finite $(N + 4k)$ dimensional Hilbert space:

$$e^{-S_b} = \text{Tr}_{(N+4k) \times (N+4k)} \text{P exp} \left[- \int d\sigma (M_0)^2 \right], \quad M_0 \equiv \begin{pmatrix} 0 & T(x)^\dagger \\ T(x) & 0 \end{pmatrix}. \quad (3.18)$$

We may substitute the tachyon configuration (3.5) to represent the system of D0-D4-branes. Note that the system is finite in the sense that the Hilbert space is finite dimensional. This is in contrast to the situation we found in the derivation of the Nahm construction of monopoles [21].

Here x^μ in $T(x)$ is considered as the world sheet string coordinate $x^\mu(\sigma)$. This $x^\mu(\sigma)$ has another important interpretation: a time dependent external field in the quantum mechanics governed by the Hamiltonian $(M_0)^2$, where σ is the (Euclidean) time of the quantum mechanical system. With this in mind, let us consider the tachyon condensation $u \rightarrow \infty$. We can diagonalize M_0 as in Sec. 3.1 by the matrices U_i , which depend on $x(\sigma)$. In the $u \rightarrow \infty$ limit, we find that the remaining terms in the path-ordered trace is just the $N \times N$ part, because the diagonalized M_0 has only N vanishing eigenvalues. The other non-zero eigenvalues give a vanishing trace due to the limit $u \rightarrow \infty$. In this “selection” of the N eigenmodes in the quantum mechanical system, note that the transformation U_i depend on σ through $x(\sigma)$. In other words, the wave function of the D-brane in the Chan-Paton space is a function of x and thus of σ . Therefore a Berry’s phase $U_i^\dagger \partial U_i$ should be associated with it. This phase is exact because the $u \rightarrow \infty$ limit is the same as the adiabatic limit in the quantum mechanics interpretation (see [21] and also [19], where the kinetic term for $x(\sigma)$ disappears in the $u \rightarrow \infty$ limit). Thus we exactly have

$$e^{-S_b} = \text{Tr}_{N \times N} \text{P exp} \left[- \int d\sigma A_\mu(x) \partial_\sigma x^\mu(\sigma) \right]. \quad (3.19)$$

This is the boundary perturbation for the N D4-branes with the gauge field A_μ , on the boundary state.

We have shown that the boundary state of the D0-D4 system (3.18) is identical with the boundary state of the D4-branes with instanton gauge field (3.19). This proves that the procedures of the tachyon condensation in Sec. 3.1 is valid in string theory. An important point is that this also gives a strong evidence that the instanton configurations on the D4-branes do not receive α' corrections. Basically there have been no reason to believe that small instantons, where the scale of the instanton gets small and the curvature is not slowly varying, do not receive any stringy corrections of α' . But here we have shown that the small instantons singularity limit of the self-dual configuration corresponding to vanishing S is string-theoretically *equivalent* to the D0-brane description and thus provides a worldsheet conformal point.

4. Inverse ADHM and Completeness from D-branes

The inverse ADHM construction is somewhat mysterious from the view point of obtaining induced connections on a sub-manifold. However the benefit of considering the ADHM construction in terms of D-branes is that also this inverse procedure is easily derived, owing to the democratic nature of D-branes. Instead of using the D-brane descent relations for the tachyon condensation, here we use the D-brane ascent relation found in [11] and developed in [18, 19]. As we shall see, it realizes the inverse ADHM construction, and the philosophy is depicted in Fig. 2.

The power of the ADHM / inverse ADHM constructions is the uniqueness and the completeness. For each instanton solutions there is a corresponding ADHM data, and vice versa. This was explicitly shown [17] by applying the ADHM and the inverse ADHM procedure succeedingly. Our D-brane realization enables one to access this completeness much more easily: we can show directly that D0-brane configurations used in the ADHM and the inverse ADHM are the same, which gives a direct proof of the completeness without referring explicitly to the procedures of the ADHM / inverse ADHM constructions.

4.1 Derivation of inverse ADHM construction

We start with giving a brief summary of the inverse ADHM construction for a reference. The inverse ADHM construction is a way to get the original ADHM data (X^μ, S) from a given instanton configuration $A_\mu(x)$. One starts with computing normalized Dirac zeromodes,

$$D^\dagger \psi = 0, \quad D \equiv e_\mu D_\mu, \quad D_\mu \equiv \partial_\mu + A_\mu. \quad (4.1)$$

Note that $D^\dagger = e_\mu^\dagger D_\mu$ has two spinor indices and accordingly ψ has a spinor index $\alpha = 1, 2$ which we often omit. Then Atiyah-Singer index theorem [25] ensures that there are k normalizable zeromodes $\psi_i(x)$ labeled by $i = 1, 2, \dots, k$ satisfying

$$\int d^4x \psi_i^\dagger \psi_j = [\mathbb{I}_k]_{ij}. \quad (4.2)$$

Usually the Dirac operator is defined as $\gamma_\mu D_\mu$ where $\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix}$, but here we call the chiral decomposed operator as a Dirac operator. The zeromode ψ has negative chirality, while there is no normalizable zeromode of D (which has positive chirality).

Let us consider some non-normalizable scalar zeromodes of the Laplacian $D_\mu D_\mu$ (note that for self-dual gauge fields we have a relation $e_\mu^\dagger D_\mu e_\nu D_\nu = D_\mu D_\mu \otimes \mathbb{I}_2$), which is going to be another important ingredient in the inverse ADHM construction. There are N non-normalizable zeromodes $\phi_a(x)$ ($a = 1, 2, \dots, N$). When there is no instanton background, these reduce to constant wave functions. The normalization of ϕ is determined in such a way that in the asymptotic region $x^2 \gg 1$ they

coincide with the original constant wave functions up to a certain $SU(N)$ gauge transformation. Since the instantons are localized near the origin, in the asymptotic region the instanton gauge field should be written as a pure gauge $A_\mu \sim g^\dagger \partial_\mu g$. Thus, if we align the N zeromodes to form an $N \times N$ matrix, it coincides with g^\dagger times the original constant wave functions.

Using these spinors and scalar zeromodes, one can reconstruct the ADHM data by the following formulas,

$$[X_\mu]_{ij} = \int d^4x \, \psi_i^\dagger x_\mu \psi_j , \quad (4.3)$$

$$[S]_{ia\alpha} = \frac{1}{2\pi} \int d^4x \, [\psi_i^\dagger e_\mu]_\alpha D_\mu \phi_a . \quad (4.4)$$

The second relation is not the familiar one written in [17] but this is the original one which can be found for example in [40]. This expression turns out to be closely related to our D-brane derivation.

Stringy derivation of this inverse ADHM construction is just the realization of the D0-D4 system in terms of infinite number of D0-branes and anti-D0-branes (see Fig. 2). According to the BSFT, D4-branes with nontrivial gauge fields on them are realized by a tachyon condensation of infinite number of pairs of D0-branes and anti-D0-branes. The precise and exact field profiles on those D0-branes are the following tachyon condensation and the transverse scalar field [11, 18, 19]*

$$\begin{aligned} T &= \lim_{u \rightarrow \infty} u(\hat{p}_\mu \otimes \mathbb{1}_N - iA_\mu(\hat{x})) \otimes e_\mu^\dagger = -i \lim_{u \rightarrow \infty} uD^\dagger , \\ \Phi_\mu^{\text{D0}} &= \Phi_\mu^{\text{antiD0}} = \hat{x}_\mu \otimes \mathbb{1}_N \otimes \mathbb{1}_2 . \end{aligned} \quad (4.6)$$

Note that \hat{p} and \hat{x} are infinite dimensional matrix representation of the Heisenberg algebra, $[\hat{x}, \hat{p}] = i$, and thus the Dirac operator is in an infinite dimensional matrix representation. In the limit $u \rightarrow \infty$ one can show via the boundary state formalism that this tachyon configuration is exactly equivalent to the D4-brane configuration with the gauge field $A_\mu(x)$.

At this stage, it is already clear that the tachyon is in fact the Dirac operator in the inverse ADHM construction. We apply the philosophy of the tachyon condensation in which only the Chan-Paton indices with zero tachyon eigenvalues survive in the $u \rightarrow \infty$ limit [29]. Then what is important is the explicit zeromodes of the tachyon T ,

$$\left[(\hat{p}_\mu \otimes \mathbb{1}_N - iA_\mu(\hat{x})) \otimes e_\mu^\dagger \right] |\psi\rangle = 0 . \quad (4.7)$$

*One may write this set as a superconnection in which the gauge transformation property is easy to read,

$$\begin{pmatrix} \hat{x}_i & T^\dagger \\ T & \hat{x}_i \end{pmatrix} . \quad (4.5)$$

Note that the zeomodes of T correspond to D0-branes and those of T^\dagger correspond to anti-D0-branes. As usual in quantum mechanics, inserting a complete set

$$\int d^4x |x\rangle\langle x| = \mathbf{1}_\infty \quad (4.8)$$

where $|x\rangle = |x_1, x_2, x_3, x_4\rangle$ is the eigen vector of the matrix \hat{x} ,

$$\hat{x}_\mu |x_1, x_2, x_3, x_4\rangle = x_\mu |x_1, x_2, x_3, x_4\rangle, \quad (4.9)$$

we recover the relation (4.1) with the definition $\psi(x) = \langle x|\psi\rangle$. (In this section, for notational simplicity, we sometimes omit the spinor and the $U(N)$ indices, which are not relevant below.) Furthermore, the normalization of the infinite dimensional vector $|\psi\rangle$ is given by $\langle\psi|\psi\rangle = 1$ which is, by again inserting the complete set (4.8), shown to be equivalent to (4.2).

On the off-shell boundary states (or the BSFT), the worldsheet boundary interaction appears in the form (3.18), and thus in effect the tachyon always appears as a combination TT^\dagger or $T^\dagger T$. So the important is the zeromodes (with positive chirality) $|\phi\rangle$ of $TT^\dagger = u^2 D_\mu D_\mu \otimes \mathbf{1}_2$ in this sense. (There may be non-normalizable zeromodes of $T^\dagger T$ with negative chirality.) Since the Dirac operator is written by an infinite dimensional matrix, the non-normalizable zeromode $|\phi\rangle$ is an infinite dimensional vector. In the absence of the gauge fields, this non-normalizable zeromode is just a vector state $|p=0\rangle$ (with the spinor and gauge indices) in the expression of the momentum eigenstates. The reason is that when $A_\mu = 0$, the BSFT tachyon potential is just $e^{-u^2 p^2}$ which, in the limit $u \rightarrow \infty$, removes all the momentum states except the zeromodes.[†] This $|p=0\rangle$ corresponds to a D4-brane. Here the normalization was fixed as usual. In the x representation, these non-normalizable zeromodes are just constant. When instanton gauge fields are turned on, we will have $2N$ non-normalizable zeromodes of TT^\dagger with positive chirality, $|\phi\rangle_{a,j}$ where $a = 1, 2, \dots, N$ and $j = 1, 2$. The index a is for the $SU(N)$ gauge group, and j is trivially related to the spinor index because TT^\dagger is proportional to $\mathbf{1}_2$. Using the scalar $\phi_a(x)$, these are written as $\langle x|\phi\rangle_{a,j} = \phi_a(x) \otimes c_j$ where c_j are constant spinors, $c_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The overall normalization should be defined such that for $|x| \gg 1$ they represent D4-branes, namely $|\phi\rangle \sim |p=0\rangle$ up to a gauge transformation. This is a normalization similar to $\phi_a(x)$ in the inverse ADHM construction.

Knowing the zeromode expressions, we proceed to get the information on the surviving D-branes. There are two kinds of D-branes surviving, corresponding to the fact that we have normalizable and non-normalizable zeromodes of the tachyon

[†]More precisely, since the Dirac operator has non-normalizable zeromodes and then has a continuous spectrum near the zeromodes, we should keep non-zeromodes which are very close to the zeromodes. This is because the D4-branes can not be described by D0-branes only, in this situation.

field: the Chan-Paton state $|\psi\rangle$ signals the surviving k D0-branes, and $|\phi\rangle$ shows the creation of the N D4-branes.[‡]

The location of the surviving k D0-branes is easily found by taking the expectation value of the original scalar field $\Phi_\mu = \hat{x}_\mu$,

$$X_\mu = \langle\psi|\Phi_\mu|\psi\rangle = \int d^4x \langle\psi|x\rangle x_\mu \langle x|\psi\rangle \quad (4.10)$$

which is in fact one of the the inverse ADHM formulas, (4.3).

Another ADHM data S should be seen from the D0-D4-string. Remember that the D0-D4-string is encoded in the tachyon field in the ADHM construction in Sec. 2.2. Since we want the D0-D4-string, what we need is the matrix transition element of the tachyon between the Chan-Paton states representing the D0-branes and the D4-branes. In fact, the matrix element of the normalizable and the non-normalizable zero modes gives

$$2\pi u S = i\langle\psi|T^\dagger|\phi\rangle = u \int d^4x \langle\psi|x\rangle D\langle x|\phi\rangle \quad (4.11)$$

which is nothing but another inverse ADHM formula, (4.4). Note that for the matrix element of T , we have no normalizable zeromode with positive chirality, and thus the expectation value vanishes. (One might think that $\langle\phi|T|\psi\rangle = 0$ implies $\langle\psi|T^\dagger|\phi\rangle = 0$, thus contradicts (4.11). However, strictly speaking, the non-normalizable modes does not reside in the Hilbert space. In appendix A, we will justify (4.11) by compactifying the R^4 to S^4 and then taking the decompactification limit.)

In this subsection we have derived the inverse ADHM construction from the D-brane ascent relation in the tachyon condensation. The important point here is that we have two kinds of D-branes surviving the tachyon condensation, and the normalizability of the zeromodes directly corresponds to the dimensionality of the remaining D-branes.

4.2 Direct completeness in terms of D-branes

The completeness basically means that the ADHM data appearing in (2.1) is identical with the data obtained by the inverse ADHM construction (4.3) and (4.4), once in the inverse ADHM construction one uses the gauge fields derived by the ADHM construction. We have already given the D-brane realization of these constructions. In this subsection we further provide a direct way how we can see that those data are the same, by using again a tachyon condensation.

The ADHM construction is realized as representing the D0-D4 system by the D4-anti-D4-branes, while the inverse ADHM construction uses the D0-anti-D0-branes.

[‡]Note that each non-normalizable zeromode doesn't correspond to a single D4-brane. In the correspondence the spinor structure of the $SU(2)$ indices doesn't count.

Therefore a direct relation between these should be seen by representing the D4-anti-D4-branes by infinite number of the D0-anti-D0-branes. See Fig. 3. The way to construct a single D4-brane out of infinite number of D0-anti-D0-branes is already described in the previous subsection, so we just do the same for all the D4-anti-D4-branes. Then the resultant D0-anti-D0-brane configuration is as follows. We have the transverse scalar field $\Phi_\mu^{\text{D0}} = \Phi_\mu^{\text{antiD0}} = \hat{x}_\mu$ as before, as well as the tachyon profile

$$T = \lim_{v \rightarrow \infty} \left(\begin{array}{ccc} \overbrace{\left(\begin{array}{cc} v\hat{p}_\mu \otimes \mathbf{1}_N \otimes e_\mu^\dagger & 0 \\ 0 & v\hat{p}_\mu \otimes \mathbf{1}_{2k} \otimes e_\mu^\dagger \end{array} \right)}^{2N\infty} & \overbrace{\left(\begin{array}{cc} 0 & \mathbf{1}_\infty \otimes uS \otimes \mathbf{1}_2 \\ v\hat{p}_\mu \otimes \mathbf{1}_{2k} \otimes e_\mu^\dagger & t^\dagger(\hat{x}) \otimes \mathbf{1}_2 \end{array} \right)}^{4k\infty} & \overbrace{\left(\begin{array}{cc} \mathbf{1}_\infty \otimes uS^\dagger \otimes \mathbf{1}_2 & t(\hat{x}) \otimes \mathbf{1}_2 \\ v\hat{p}_\mu \otimes \mathbf{1}_{2k} \otimes e_\mu & v\hat{p}_\mu \otimes \mathbf{1}_{2k} \otimes e_\mu \end{array} \right)}^{4k\infty} \end{array} \right) \left. \begin{array}{l} \} 2N\infty \\ \} 4k\infty \\ \} 4k\infty \end{array} \right. . \quad (4.12)$$

The entries including v give rise to the D4-branes and the anti-D4-branes in the limit $v \rightarrow \infty$. The upper-left $(2N + 4k)\infty \times (2N + 4k)\infty$ matrix corresponds to the $(N + 2k)$ D4-branes, while the lower-right part is for the $2k$ anti-D4-branes. The tachyon configuration (3.5) in the resulting D4-anti-D4-brane appears in the off-diagonal part of the total tachyon matrix.[§] So if we take $v \rightarrow \infty$ limit first, then we end up with the D4-anti-D4-brane configuration and goes back to the starting point of Fig. 1, *i.e.* the D0-brane point of view for the ADHM construction.

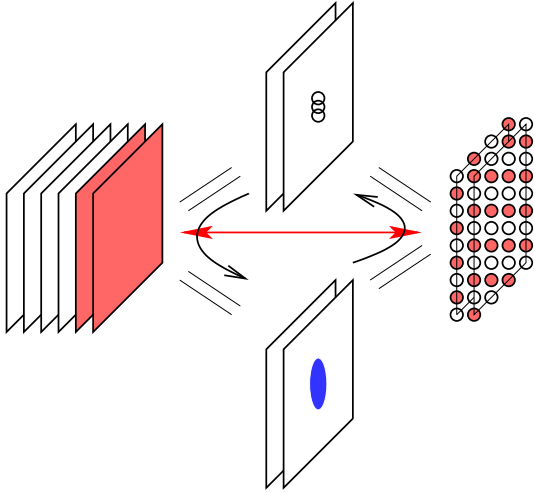


Figure 3: The D-brane realization of the completeness. The completeness and the uniqueness are equivalent to the fact that this circle with four corners is in fact closed. The horizontal arrow is a short-cut, which is provided in D-brane language and is a proof of the completeness.

On the other hand, we can take $u \rightarrow \infty$ limit first. Then, we can use the same gauge transformation U_1 and U_2 in Sec. 3.1 (but the argument x replaced with \hat{x}) to diagonalize the S and t part of the matrix (4.12). With this gauge transformation, the upper-left \hat{p}_μ is transformed by U_1 to $\hat{p}_\mu - iU_1^\dagger \partial_\mu U_1$. Note that, the part of the Chan-Paton indices which represent the $2k$ pairs of D4-anti-D4-branes will drop by the tachyon condensation in the limit, and therefore only the upper-left corner survives. Finally we get the D0-anti-D0 system with (4.6) where $A_\mu(x)$ is given by the ADHM construction from the ADHM data S, X appearing in (4.12). This means that the

[§]Here we identified the off-diagonal elements in (4.12), t and S^\dagger , as the tachyon of the D4-anti-D4-branes. We easily see that this identification is correct, by using the Gamma matrix representation when the number of the D4-branes is the same as that of the anti-D4-branes [19]. Note that the T^\dagger contains t, t^\dagger, S and S^\dagger , so the chirality operators are different in the D4-brane and the D0-brane pictures. Actually, an oriented open string connecting a D4-brane and an anti-D4brane is composed of the ones connecting D0-branes and anti-D0-branes with both orientations.

D-brane system considered in Sec. 3.1 and Sec. 4.1 are indeed the same.[¶]

We shall proceed to show the “direct” completeness hidden in the big tachyon matrix (4.12). We consider the same limit as above, $u \rightarrow \infty$ first and then $v \rightarrow \infty$, but look at only a part of the tachyon matrix (4.12) — the condensation of the lower-right $4k\infty \times 4k\infty$ corner, and diagonalize it first. In the limit $u \rightarrow \infty$, it is easy to show that there appear k zeromodes in this part of the tachyon matrix. These should correspond to the remaining k D0-branes. When X_μ in t is simultaneously diagonalizable, it is obvious that we get X_μ as the location of the resulting k D0-branes. On the other hand, in the inverse ADHM construction of Sec. 4.1, the data X_μ is given by the location of the k D0-branes, so we are dealing with the same physical quantity here. Therefore we could directly show that X_μ in the tachyon profile of (3.5) is identical with that of (3.16). This is the completeness.

Let us also derive the completeness for S . For simplicity we put $X_\mu = 0$ in the following. After this “partial” tachyon condensation, we may neglect the vanishing pairs of the D0-branes and anti-D0-branes and deal with only the surviving k D0-branes. Using this new and reduced number of basis, one can show that the above matrix can be represented as^{||}

$$T = \lim_{v \rightarrow \infty} \left(\overbrace{v\hat{p}_\mu \otimes \mathbb{1}_N \otimes e_\mu}^{2N\infty} \overbrace{2\pi v S|x=0\rangle}^k \right) \Big\}^{2N\infty}. \quad (4.13)$$

[¶]In particular, if we use the completeness found in [17], we conclude that the hypermultiplet S appearing in (3.5) is identical with S in (4.11).

^{||}Here we demonstrate how we obtained the matrix element $vS|x=0\rangle$ in (4.13) briefly. The $4k\infty \times 4k\infty$ corner of the matrix (4.12) simply states a sequence of the tachyon condensation, $4k\infty$ D0-anti-D0 \rightarrow $2k$ D4-anti-D4 \rightarrow k D0. To avoid the complicated matrix structure of (4.12), we consider a simplified sequence 2∞ D0-anti-D0 \rightarrow D2-anti-D2 \rightarrow D0, whose tachyon matrix \tilde{t} and its normalizable zeromode $\tilde{\psi}(x) = \langle x|\tilde{\psi}\rangle$ are given by

$$\tilde{t} = \begin{pmatrix} v(\hat{p}_1 + i\hat{p}_2) & u(\hat{x}_2 - i\hat{x}_1) \\ u(-\hat{x}_2 - i\hat{x}_1) & v(\hat{p}_1 - i\hat{p}_2) \end{pmatrix}, \quad \tilde{\psi}(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sqrt{\frac{u}{2v}} \exp\left[-\frac{u}{2\pi v}(x_1^2 + x_2^2)\right].$$

The zeromode wave function becomes $\sqrt{\delta(x)}$ in the $u \rightarrow \infty$ limit. An analog of this wave function in our precise 4 dimensional case for the part of (4.12) is given by $\langle x|\tilde{\psi}\rangle = \tilde{\psi}_0(u/2\pi v) \exp[-(u/2v)r^2]$ for $k=1$, where $r^2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $\tilde{\psi}_0 \equiv (i, 0, 0, i, 1, 0, 0, 1)^T$. Then, the matrix element which we want to evaluate, among the $2N\infty \times 8k\infty$ upper-right corner of the matrix (4.12), is just

$$\begin{aligned} u \begin{pmatrix} 0 & 0 & 0 & 0 & S_1 & S_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & S_1 & S_2 \end{pmatrix} |\psi\rangle &= \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \int d^4x |x\rangle u \frac{u}{2\pi v} \exp\left[-\frac{u}{2v}r^2\right] \\ &= \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \int d^4x |x\rangle 2\pi v \left(\frac{u}{2\pi v}\right)^2 \exp\left[-\frac{u}{2v}r^2\right] \xrightarrow{u \rightarrow \infty} 2\pi v S|x=0\rangle. \end{aligned}$$

Although the procedure of diagonalizing a part of the tachyon matrix (4.12) first seems not appropriate, the resultant reduced tachyon matrix (4.13) has no dependence on u that justifies the partial diagonalization.

We see that ADHM data S is appearing in the tachyon matrix, in such a way that it is a matrix element of k normalizable modes and N non-normalizable modes. This shows that S is the one given in (4.11), and the completeness is proven. In the matrix, $|x = 0\rangle$ state is appearing as a coefficient of S , which reflects the fact that the normalizable zeromode wave functions are localized at the location of the k D0-branes. Note that it appears as $2\pi v S$ which is the correct normalization in view of (4.6) and (4.11) with replacing u by v there.

5. Conclusions and Discussions

In this paper, we have derived the ADHM construction of instantons in string theory. The ADHM procedures appear as a selection process of remaining D-branes in the tachyon condensation which unifies the D0-branes and D4-branes. The physical meaning of the ADHM procedures are found as follows:

- The Dirac operator (2.1) in the ADHM construction is the tachyon connecting $N + 2k$ D4-branes and $2k$ anti-D4-branes.
- The zeromodes of the Dirac operator (2.1) is the Chan-Paton wave function of the D4-branes surviving the tachyon condensation.
- The ADHM formula (2.2) is the connection induced by the basis change of the Chan-Paton space, looked by the remaining D4-branes. It can be viewed also as a Berry's connection on the boundary state.

For the inverse ADHM construction, we used the D-brane ascent relation for the relevant tachyon condensation, and the inverse ADHM formulas (4.3) (4.4) turned out to be the vacuum expectation values of the Higgs and the tachyon fields of the system of infinite number of D0-anti-D0-branes. We have demonstrated that the completeness can be shown easily in the D-brane setup, and the derivation of the ADHM construction in noncommutative space was given.

As emphasized in the introduction, the equivalence of the gauge configurations on the D4-branes and the D0-D4 bound state is quite nontrivial. Let us say more concretely on this by looking at the low energy limits of the two descriptions. For simplicity we consider a single instanton in $SU(2)$ Yang-Mills theory. The size of the instanton ρ is proportional to S . If we have normalized S such that it has a scale of length, terms of higher order in S would dominate the effective action of the D0-branes in the low energy limit $\alpha' \rightarrow 0$ with S kept finite, and this means it would not be an appropriate action in the limit. Thus naturally S has mass dimension one (or at least positive mass dimension), and so $\rho = \alpha' S$. The instanton picture is natural in the $\alpha' \rightarrow 0$ limit with finite instanton size since there the Yang-Mills action is trustable. On the other hand, the D0-brane picture, *i.e.* the action using

S and X , is valid and natural in the zero slope limit with S kept finite. Therefore these two pictures reside in different regimes of the validity.

Nevertheless, we have shown that these two pictures are equivalent at off-shell. On-shell configurations is given by imposing the supersymmetry conditions on the boundary state, and in our formalism, at the opposite end points of the moduli space evidently we can get the familiar BPS conditions by imposing the supersymmetry conditions: at $\rho \ll \sqrt{\alpha'}$ we obtain the ADHM equation for S and X^μ , while at $\rho \gg \sqrt{\alpha'}$ we obtain the instanton equation. Let us consider the former region where the description by the ADHM data is natural. We have shown that the boundary state at this parameter region is equivalent to the D4-brane boundary state with the instanton gauge field constructed by the ADHM construction. But in this latter description the equations of motion for the gauge field is quite complicated with all order α' corrections. This proves that the small instanton configuration of sub-stringy size is protected against the α' corrections.

In the middle of the moduli space, since we don't know how the supersymmetry condition works explicitly, we cannot give any concrete result. However, at the both ends of the moduli space, $\rho \gg \sqrt{\alpha'}$ and $\rho \ll \sqrt{\alpha'}$, the self-dual configuration solves the BPS equation. Thus we reach the conjecture stating that *self-dual configurations with arbitrary size of the instanton solve the equations of motion of Yang-Mills field corrected in all order in α' in string theory*. In other words, self-dual configurations are solutions of non-Abelian Born-Infeld theory with higher derivative corrections in string theory.

As an affirmative evidence for this conjecture, we note the following fact. The leading α' correction to the Yang-Mills action on the D4-branes [41] is given by the first nontrivial terms in the expansion of the non-Abelian Born-Infeld action defined with the symmetrized trace [42]. In fact, it has been shown that the self-dual configuration solves the equations of motion of that theory [24]. So this is the evidence for the conjecture. At the next order $(\alpha')^3$, it has been known that the corrections differ from the terms of the action given by the symmetrized trace [43], and their explicit expression was computed in [44, 45] and has been successfully tested in [46]. Even at this $(\alpha')^3$ order the self-dual configuration solves the equations of motion [47]. (At this order BPS equations in higher dimensions generically have correction terms, but only in 4 dimensions they vanish.) It would be interesting to check that the correction terms of the order $(\alpha')^4$ computed in [44, 48] (and tested in [49]) may not modify the self-dual configuration. Note that this is the evidence in the α' expansions of the instantons, and in our paper we give the evidence for the opposite side of the moduli space of the instanton, that is, small instantons. These together suggests strongly that the conjecture is true.*

In this regard, the results of the present paper ensures that D-brane techniques

*There is a subtlety concerning field redefinitions. Since the self-dual equation is not invariant under some field redefinitions, this conjecture is true only for some particular definition of fields,

for field theory solitons are trustable, in spite of the difference in the validity of the regions in switching from one description to the other. The implementation of the solitons by the tachyon condensation provides a new understanding of the mysterious ADHM / Nahm constructions, and opens up new possibilities to view various other solitons in a different and unified manner.

Several discussions related to the results of this paper are in order.

- Atiyah-Singer index theorem [25]. The index theorem is obtained in K-matrix theory [19]. The physical equivalence between Dp -branes and pairs of D0-anti-D0-brane can be considered as a generalization of the index theorem. (It is related to the topological and analytic K-homologies, the KK-theory, the family index theorem and the Connes's spectral triple.) There the important ingredient is the boundary perturbation, which can be expressed as a quantum mechanical partition function. Evaluating it for D0-brane charge, namely taking the overlap of the boundary state and the Ramond-Ramond state, we have the index theorem. The ADHM construction has close relation to the index theorem, however, it is not just topological. As we have seen in Sec. 4.1, the field profile (4.6) gives the ADHM construction in D0-anti-D0-branes and the corresponding boundary perturbation is the same one used to show the index theorem. Thus we can say that the boundary perturbation unifies those. The D4-brane (or a topological picture) is obtained in a path-integral representation of the quantum mechanical partition function in the boundary perturbation while the D0-brane (or an analytic picture) is obtained in an operator formalism of it.
- Evaluation of the action of the BSFT. In this paper we presented the off-shell boundary state, and in principle the overlap $\langle 0|B \rangle$ provides the worldsheet partition function which is the boundary superstring field theory action. For the tachyon configuration given in this paper, it would be possible to compute this action explicitly at least as a perturbation in terms of S . The resultant equations of motion should be consistent with the ADHM equation. This explicit check and possible all-order computation in S would help the understanding of the BPS nature in the middle of the instanton moduli space.
- Octonionic instantons [50] / higher dimensional generalizations of the ADHM construction. A generalized ADHM construction in $4n$ dimensions has been proposed [51], while it is not clear how this can be embedded in string theory. At least for $n = 2$ ($d = 8$), it might be related to D0-D8 bound states [52] for

although at the end points of the moduli space the field redefinitions do not change the BPS conditions because the redefinitions are α' corrections. This field redefinition subtlety is related to the choice of the regularization of the world sheet theory in the BSFT or the boundary state formalism.

which one needs D8-anti-D8-branes and the appropriate tachyons analogous to our setup. We can expect that their noncommutative generalizations [53] may follow as in the present paper.

- We can repeat what we have done in this paper in type I string theory instead of type II string theory. Then we will have the ADHM construction for the SO or Sp gauge theory. It will be interesting to generalize our method to the torus case, namely the Nahm transformation. Another interesting extension of the D4-D0 bound state is the fuzzy funnel [54] which is an intersecting D1-D5 system (the intersecting D1-D3 system has been analyzed in our previous paper [21] as the Nahm's construction of monopoles). In this case, there is no supersymmetry, however, our method might work because we have not used the supersymmetry explicitly.

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A. Note on Non-Normalizable Zeromodes

In this appendix we consider a consistent definition of the Dirac operator to resolve the problem mentioned below the equation (4.11). The meaning of the ADHM formula (4.4) will become clearer.

We define the Dirac operator $\mathcal{D} \equiv \gamma_\mu \mathcal{D}_\mu$ which is now the usual definition in the 4×4 matrix form, and the chirality operator Γ^5 . Let us consider the non-normalizable zeromode ϕ^+ which appeared in Sec. 4.1. The superscript indicates the chirality of the spinor. Then, because of the equation $(D_\mu D_\mu \otimes \mathbb{1}_2) \phi^+ = 0$, $v^- \equiv D \phi^+$ is a zeromode of D^\dagger , *i.e.* $D^\dagger v^- = 0$. Due to the asymptotic behavior of ϕ^+ , we can see that v^- is normalizable and then can be written as a linear combination of ψ^- . Therefore, we obtain a simple relation $v^- = S \psi^-$ where S is the $N \times 2k$ matrix which acts on the spinor index and also on the index labelling the k normalizable zeromodes.

Note that in R^4 , the zeromode of \mathcal{D}^2 is not necessarily a zeromode of \mathcal{D} . Thus if we include the non-normalizable modes, \mathcal{D} is not Hermitian and it is unclear how to define \mathcal{D} . This is the problem which we mentioned below the equation (4.11). To overcome this problem, we consider an S^4 and taking a limit of large radius to R^4 . In this way, we can justify the use of the non-normalizable modes, and \mathcal{D} is now defined as a Hermitian operator. In a compact space, an eigen state of \mathcal{D}^2 , $\mathcal{D}^2|m^2\rangle = m^2|m^2\rangle$, can be decomposed as $|m^2\rangle = |m\rangle + |-m\rangle$ for $m \neq 0$ where $\mathcal{D}|\pm m\rangle = \pm m|\pm m\rangle$ and $\Gamma^5|m\rangle = |-m\rangle$. Then any zeromode of \mathcal{D}^2 is a zeromode of \mathcal{D} . This seems to be contradicting the R^4 case. However, we will see it is not, by looking at the large radius limit carefully below.

Consider an S^4 with a very large radius. Then there are eigenmodes whose eigenvalues are zero or very close to zero which correspond to ψ^- and ϕ^+ in the large radius limit. Suppose that η which is not a zeromode corresponds to the ϕ^+ . Then we have

$$(\mathcal{D} + m)\eta = 0 . \quad (\text{A.1})$$

And this is written in the chiral decomposed form as

$$\begin{pmatrix} m\mathbb{1}_2 & D \\ D^\dagger & m\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} \eta^- \\ \eta^+ \end{pmatrix} = 0 , \quad (\text{A.2})$$

where in the limit to R^4 the mass parameter m is taken to 0. This equation means

$$(D^\dagger D - m^2)\eta^+ = 0 , \quad \eta^- = -\frac{1}{m}D\eta^+ . \quad (\text{A.3})$$

Since ϕ^+ is non-normalizable and we supposed $\eta^+ = C\phi^+$ in the limit, the normalization constant C appearing here should go to zero in the limit. (Otherwise the above equation wouldn't make sense.) Thus, in the large radius limit $m \rightarrow 0$, $\frac{C}{m}$ will be kept finite and η^- should be a linear combination of ψ^- since it is normalizable. In this way we can have a consistent result,

$$D^\dagger D\phi^+ = 0 , \quad S\psi^- = D\phi^+ . \quad (\text{A.4})$$

Here the important point is that the usual normalization of the state in S^4 is different from the plane wave normalization of the state in R^4 .

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